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ABSTRACT

This paper examines the category-theoretic formulation of cognitive development introduced by Piaget in the late 1960's and elaborated during the 1970's. The new theory is interpreted as the focal point of Piaget's investigations into topics such as function, correspondences, and commutability. Hypotheses arising from Piaget's new model were investigated in a developmental study of the morphism concept, which is central to the category-theoretic formulation. Subjects, 36 boys and 36 girls ranging in age from 5 to 7 years, were tested individually on three morphism problems. Each problem consisted of a game board divided into colored sections, a number of colored wooden tiles to be mapped to these sections, and a picture representing an operation or relation to be preserved while conducting the mapping. Results, as hypothesized, indicated that cognition of morphisms accompanies or precedes operational reasoning. Discussion focuses on several implications of further developing Piaget's category-theoretic model. Directions such development might take are pointed out. It is concluded that the new formalism offers the potential for an integrated model of cognitive process and cognitive structure, and is therefore a significant advance in constructivist theory. (Author/RH)

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Development of Morphisms: An Initial Investigation and its Link with Piaget's New Theory of Concrete Operations

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Abstract

This paper examines the category-theoretic formulation of cognitive development introduced by Piaget in the late 1960's and elaborated during the 1970's. The new theory is interpreted as the focal point of Piaget's investigations into topics such as functions, correspondences, and commutability. Hypotheses arising from Piaget's new model were investigated in a developmental study of the morphism concept, which is central to the category-theoretic formulation. The results indicate that cognition of morphisms accompanies or precedes operational reasoning, as hypothesized. A number of implications of the new model and possible future directions are then explored. It is concluded that the new formalism offers the potential for an integrated model of cognitive process and cognitive structure, and is therefore a significant advance in constructivist theory.

Development of Morphisms: An Initial Investigation and its Link with Piaget's New Theory of Concrete Operations

Up to the end of his career, Piaget continued to refine his descriptions of cognitive structures and their development, and in doing so relied on ongoing work in mathematical logic and abstract algebra. A good example is his analysis of concrete operations. In his first systematic formulations, Piaget (1942, 1949) was much influenced by the Bourbaki school and its emphasis on the foundational properties of lattice and group structures. Accordingly, Piaget's fundamental operatory structures -- the groupings -- were hybridized from lattice- and group-theoretic concepts. Although the groupings successfully (if inelegantly) captured the reversible binary class and relational operations postulated for middle childhood, by the 1970's Piaget believed there was much more to the story of the origins of operations than the grouping theory could describe, and turned to a less limiting formalism: category theory (MacLane 1971).

In this paper I first give an interpretive summary of Piaget's shift toward describing cognitive development by means of category-theoretic notions, and here the qualifier "interpretive" must be emphasized. In this section I attempt to link up a number of

strands evident in Piaget's recent writings, such as the ideas of function, correspondence, commutability, and in particular the category-theoretic notion of morphism. Next I describe a developmental investigation of children's reasoning on tasks involving morphism properties. Finally, I discuss some practical and theoretical issues involved in following up Piaget's recent category-theoretic formulations.

Functions and Operations

The Genevan interest in category theory can be traced to the work by Piaget et al. (1968/1977) on functions. This corresponds to the period when American mathematicians were beginning to investigate a new set-free mathematical foundation based on category theory (e.g., Lawvere, 1966). In the function project, the primary goal was to provide a positive structural description of thought during the period of 2 to 6 years, which had earlier been described in terms of "transduction" (e.g., Piaget, 1924/1926). Piaget et al. (1968/1977) proposed that young children's thought has the structure of functions ($y=fx$), whose source is the assimilatory action sequences of sensorimotor stage 5. For example, in playing with a ball, an infant might construct the relation, "the harder I push, the further it goes;" that is, distance is a function of force. During the period of 2 to 6 years, these functional schemata become increasingly representational.

Such action-based function notions can be combined (composed), but do not in general have inverses. This is equivalent to the earlier description of young children's thought as irreversible, or preoperational, and therefore not formalizable under operational structures such as groups or groupings. The structure of young children's thought is, however, formalizable under the weaker axioms of category theory, which will be described in a subsequent section. In presenting this formal model of preoperations, Piaget et al. (1968/1977) highlight the systematic nature of children's competencies--which they describe as a "semi-logic"--rather than the inadequacies that appear in comparison with concrete operational thought. According to the 1968 account, this functional semi-logic lays the groundwork for reversible binary (i.e., grouping) operations, in as much as it provides a basis for reflective abstraction upon the organization of actions and their combinations (including, potentially, combinations of inverses).

Correspondences and Transformations

As a tool for epistemological analysis, category theory led Piaget (1977, 1979, 1980) to more explicitly distinguish between correspondences and transformations, and to assign the former a more important role in cognitive development than had been done earlier. One aspect of this distinction follows definitionally from the fact that functions are a subclass of relations, and operations are a

subclass of functions. That is, if what we mean by a transformation is represented formally by some kind of function, then clearly there exist non-transformational relations or correspondences. This definitional distinction hinges on the fact that functions are single valued (*e.g.*, in the case of a unary function f, for any pair $\langle x,y \rangle$ in f, y is unique for that x). By the requirement of a unique value, functions entail an idea of directionality which the broader class of correspondences does not; this directionality is conventionally symbolized--especially in category theory--by the arrow whose point indicates the range of a function's unique values (*e.g.*, $f:X \longrightarrow Y$). The requirement also entails that any composition gf of functions is also a function.

A second, more subtle aspect of this distinction applies within the class of functions: It is more natural to think of certain functions as signifying some sort of comparison among elements, and it is more natural to think of other functions more actively as signifying an exchange or transformation of one element into another. As a simple example, it seems natural to think of the identity function $fx = x$ as indicating pairs of identical elements; whereas, it seems more natural to think of a function such as $fx = (x + 1) - 1$ as indicating a method for computing a value for each x. Notice that this is essentially a psychological and a semantic distinction, since each of these examples is a well defined

function. In fact, both functions have the same extension, although their intensions are evidently distinct. This potential for "slippage" between ostensive and computational interpretations suggests that functions might be useful for modeling psychological coordinations between correspondences and transformations. In particular, Piaget proposed that the kind of functions studied in category theory--morphisms--are particularly suited for this purpose. It will therefore be useful to review briefly what a category is.

Morphisms and Categories

A category is a collection of mathematical objects together with a collection of morphisms defined on those objects (see MacLane, 1971, pp. 10-12, 24-26). Objects include points, sets, posets, groups, topological spaces, and so forth. Morphisms (also called arrows) are functions on the given class of objects which preserve their structure. As an intuitive example of a morphism, in Figure 1(1) **A** and **B** represent linear orders, with L_1 being the order relation for **A** and L_2 the order relation for **B**. Then, $\underline{f}$ is a morphism of order because every pair of elements (*e.g.*, <1, 2>) in the relation L_1 is mapped to elements (*e.g.*, <b, c>) that are in the relation L_2. In contrast, in Figure 1(2), $\underline{g}$ is not a morphism of order because <1, 2>, which is in L_1, is mapped to <b, a> which is not in L_2. As a second example, a morphism of groups is one that

preserves the binary operation. Figure 1(3) shows selected elements in two groups **A** and **B**, and the products of illustrative combinations (e.g., 1 + 2 = 3 in **A**). The function $\underline{h}$ is a morphism of groups since it maps the values of the operation in **A** to those of the corresponding operation in **B**: that is, $\underline{h}(\underline{m} + \underline{n}) = \underline{hm} + \underline{hn}$ for all $\underline{m}$, $\underline{n}$ in **A**.

Insert Figure 1 About Here

In its focus on morphisms, category theory is more concerned with investigating classes of functions that share correspondence properties, than with computational properties of individual functions--just as, for instance, in creating Figure 1(3), my choice of the individual function $\underline{hx} = 5\underline{x}$ is entirely arbitrary, since any number of functions would serve as well to illustrate the general idea of a morphism of groups.

We are now equipped with two intuitive examples of categories: the category **LinOrd** consisting of all linear orders together with all functions that behave like $\underline{f}$ in Figure 1; and the category **Grp** consisting of all groups together with functions that behave like $\underline{h}$. More formally, the axioms for categories are the following (cf. MacLane, 1971, pp. 7-8):

Identity. Each object $\underline{A}$ has an identity arrow $\mathrm{id}_{\underline{A}}:\underline{A}\longrightarrow\underline{A}$.

Composition. For each pair of arrows $\langle \underline{g},\underline{f} \rangle$ in which the domain

of g coincides with the codomain of f, that is, $f:A \longrightarrow B$ and $g:B \longrightarrow C$, the composite gf is defined.

Unit law. Whenever gf is defined, the identities $id_B f = f$ and $g id_B = g$ are also defined.

Associativity. Whenever the composition of arrows h,g,f is defined, $h(gf) = (hg)f$.

Of course, the idea of morphism occurs in many branches of mathematics (e.g., homeomorphisms in topology) and did not originate with category theory. However, traditional definitions of morphisms rest on set-theoretic foundations, just as for expository purposes I have so far described categories in set-theoretic terms (objects are sets with structure, morphisms are defined on objects, etc.). The innovation of category theory is that categories can also be defined axiomatically independent of set theory. This should make sense intuitively if we consider that, since each object can be identified with its identity arrow (see above axioms), it is possible to dispense with objects (e.g., sets and elements) entirely and develop the theory from axioms taking arrows as primitive terms. Indeed, part of the interest in category theory among mathematicians is its potential for providing a set-free form of mathematical foundations (Lawvere, 1964, 1966).

Such considerations apparently led Piaget to hypothesize that the concept of morphism, or structural correspondence, may be

epistemologically foundational rather than derivative. In the study of cognitive development, this translates into the hypothesis that the activity of constructing correspondences (comparisons, analogies, etc.) may be independent of, and of equal importance to, the activity of evaluating transformations (Piaget, 1979). This is in contrast to his earlier position, according to which correspondences arise either incidentally or as a derivative of the child's investigations of transformations, the latter providing most of the developmental impetus.

As an example of the new emphasis on correspondences, we may consider the interpretation of conservation in terms of commutability (Inhelder, Blanchet, Sinclair & Piaget, 1975; Piaget, 1974/1980, chap. 11). Conservation is the inference of invariant quantity under a non-quantitative transformation. Commutability is the inference that in such a transformation, the part of an object that is removed (*e.g.*, in width) corresponds exactly to the part that is added (*e.g.*, in length). Figure 2 illustrates this correspondence in the case of continuous substance. Here, the object is conceived as having an internal structure consisting of some relation among the object's parts (*e.g.*, $\{\langle \underline{a},\underline{a'}\rangle,\langle \underline{b},\underline{b'}\rangle,\ldots\}$). Conservation is the construction of a morphism representing some correspondence between the internal structure prior to, and after, the transformation. The experiments on commutability led to the hypothesis that "correspondences pave the way for the

transformations, since there are no transformations that the child discovers without first having made certain correspondences" (Piaget, 1979, p. 20).

Insert Figure 2 about here

Categories and Concrete Operations

Piaget (1977) proposed a new formulation of concrete operational groupings based on category theory. As was pointed out above, intermediate level competencies found between consolidated sensorimotor cognition and concrete operational cognition do not conform to the group-like formalisms that Piaget had previously relied upon. However, a category-theoretic formalism can be applied equally well to these intermediate competencies and to the more highly structured competencies of middle and late childhood. The new formulation takes account of the recent research linking functions to the emergence of operations, and of the recent emphasis on the importance of correspondences. The categorical formalism, which can encompass a wide range of different structures under a few axioms, also represents a potential increase in theoretical elegance and efficiency over the previous grouping formalism with its numerous specialized axioms. (As Piaget [1977] notes, the previous grouping model "has generated little enthusiasm from logicians and

mathematicians because of its unavoidable limitations" [p. 351]).

Piaget (1977) provides only a sketch of the new formulation. However, as one example of how this approach could be fleshed out, it would be possible to define a category **Gpment** with objects G_i the previously defined groupings, and morphisms M_i all maps that carry the internal structure of one grouping to that of another. For example, letting G_1 = the grouping for class addition, G_4 = the grouping for class multiplication, and G_5 = the grouping for serial addition, there would exist a morphism M_i, mapping class unions to class intersections, M_j, mapping class unions to serial unions, and so forth. The same approach could be extended to include both discontinuous (logical) and continuous (infralogical) structures. Notice that in this description morphisms are "figure" while intrastructural operations and relations are "ground;" that is, the latter are implicit in the former. Cognitive activity in such a system could be described in terms of assimilations of observables to a given morphism, with transformations entailed by that morphism being carried out implicitly. An example might be a subject in a conservation experiment who immediately infers that quantity is conserved by assimilating the problem to a subdivision-preserving morphism (i.e., creating a correspondence between the particular task and a generalized schema for commutability); the necessary inverse and reciprocal operations, being implicit in the original construction of the commutability morphism, may play no conscious

role in the subject's spontaneous inference. (Notice that assimilation itself is here interpreted as a morphism between an "underlying set" of observables and a conceptual structure; this is elaborated further in the discussion section, under "process theory.")

It remains to be seen whether a category-theoretic formalization of concrete operations and beyond can be worked out in detail. Such a project is subject to numerous constraints, including not only the requirement of consistency with available empirical data, but also the requirement of consistency with an appropriate process model of cognition such as, for instance, equilibration theory (Piaget, 1975/1986). Nevertheless, even at its current state of elaboration, one notes the increased power of a formal theory that puts cognition from infancy through adulthood in reach of a single formalism.

Focus of the Present Research

The primary line of evidence that operational thought involves the cognition of morphisms rests on conceptual analyses by which previously identified class, relational, and quantitative abilities are reinterpreted in terms of morphisms. For example, Piaget (1977) interprets the vicariance operation (Grouping II) as a rearrangement morphism on classes that preserves class extension (see Figure 2). The purpose of the present research was to investigate the

development of morphism abilities more directly, through the use of tasks that are explicitly designed to embody morphism properties. Based on the category-theoretic interpretation of concrete operations, it was hypothesized that the ability to preserve operational or relational structures while making a correspondence would increase during the period of 5 to 7 years.

Subjects and Procedure

The subjects, 36 boys and 36 girls ranging in age from 5 to 7 years, were tested individually on three morphism problems.

Each problem consisted of a game board divided into colored sections, a number of colored wooden tiles to be mapped to these sections, and a picture representing an operation or relation to be preserved while conducting the mapping (see Figure 3). Each tile had a primary color, as well as portions of other colors around its border. The primary color was used in mapping the tile to the game board. The secondary colors on the tiles were used to maintain the rules of an operation or relation.

Insert Figure 3 about here

As shown in Figure 3(1), Task 1 required mapping red and green squares to the corresponding squares on the board, and also obeying the following rules about how adjacent pairs of tiles should be

oriented: Red + Red = Green; Green + Green = Green; Red + Green = Red. These binary combinations in fact constitute the multiplication table of the 2-group.

Task 2 required mapping squares of four different colors to the corresponding colors of the game board, while following a rule that any 2 adjacent squares of the same color should create a white dot at their common border. This is a subset (i.e., the main diagonal) of the multiplication table of the 4-group. (Incidentally, the board was designed so that by following this rule, all remaining binary combinations would also conform to those of the 4-group.)

Task 3 was concerned with preservation of a relation, rather than an operation. The relation, shown in Figure 3(3) is a cyclic order in which grey dots are always just left of yellow, yellow just left of blue, blue just left of grey, and so on. The order was illustrated by a picture; the children's task was to map colored triangles to corresponding colors of the game board, while orienting adjacent pairs of triangles to reproduce the pattern of dots shown in the picture.

In each task, children first practiced making allowable binary combinations on the table top, before proceeding to try and preserve these combinations under constraint of matching individual tiles to correct colors of the playing board.

Results

Three variables were coded on each task: element mapping (number of tiles correctly mapped), structure mapping (number of correct binary products), and strategy. Element mapping and structure mapping were technically independent since it was possible to obtain a perfect score on one and on a zero score on the other; in practice, however, they were highly associated. Strategies were categorized into 4 types: (1) Trial-and-error: adding pieces to the puzzle randomly and making repeated adjustments with no evidence of an overall plan; (2) Element oriented: mapping pieces to correct colors of the game board, followed by adjusting for binary combinations; (3) Structure oriented: placing correct binary combinations, followed by adjusting tiles to colors of the game board; (4) Anticipatory: systematically mapping binary pairs to correct pairs of game board colors as if following an overall plan.

Scores were then composited to yield overall element mapping and structure mapping scores, and frequency-of-use scores for each of the four strategies. Table 1 shows the mean score by age level for all variables.

Insert Table 1 About Here

Element mapping is essentially at the ceiling for all age groups, the maximum possible score being 14.67 (averaging the maxima

of the three tasks results in the non-integer maximum). Structure mapping is also within the youngest children's reach; on average they achieve 68% of the maximum possible score of 23.67. However, structure mapping performance continues to incline with age.

Most of these successful performances, it should be noted, are achieved through trial-and-error or non-coordinated strategies. Use of anticipatory strategies increases consistently with age, and begins to over-take trial-and-error by age 7.

Discussion

Consistent with the category-theoretic formulation, the present research indicates that the development of operations and relations can be observed through tasks that involve preserving given combinations under a correspondence. Piaget (1979, 1980) distinguishes five kinds of relations between correspondences and transformations, which roughly break down into three types: First, there are simple correspondences that can be performed empirically; these do not depend on understanding any transformations that may be involved, although they may facilitate such understanding by drawing attention to parallel patterns or outcomes. Second, there are mutually facilitating situations in which understanding the transformation and performing the correspondence are intimately linked. Finally, there are correspondences that arise only after a system of transformations is well conceptualized.

The present examples fall midway between the first and second types. Thus, it is possible to solve these problems empirically, through trial-and-error. In this case, subjects treat the operation to be preserved (e.g., the picture showing adjacent squares) as a figural pattern, and make ad hoc readjustments in their production as needed to match the pattern. On the other hand, subjects who cognize the operation as a system of rules proceed immediately to map binary combinations (i.e., to map structure), rather than considering each element sequentially and individually. In this case, understanding the transformation facilitates constructing the correspondence.

Using the same materials, it would be possible to study correspondences that are even more dependent on transformational cognition. Rather than performing an identity mapping as described above (red tiles to red spaces, etc.), subjects could be asked to construct some other well defined mapping (red to green, etc.) while also preserving an operational structure. Introducing aspects of transformation into the correspondence itself, or other similar task variations, would probably make empirical solutions more unwieldy and costly in terms of memory demands, and might therefore be more effective in revealing the transition from preoperational to operational cognition of morphisms.

Future Directions

In this section I examine a few implications of developing Piaget's category-theoretic model further, and suggest a few directions that such a development might take.

Measurement issues. First, it must be recognized that evaluating and fine-tuning the new model will depend on devising appropriate tasks and appropriate response criteria. It will be necessary to go beyond the strategy of trying to interpret children's performances on standard measures of conceptual ability as reflecting an underlying morphism. Rather, tasks need to be designed that embed relevant operational abilities within morphism contexts in a manner specified by the model.

A related issue is the definition of appropriate response criteria. Because of the desirability of investigating synchronies and acquisition sequences among distinct concepts, response criteria need to be defined in a principled manner rather than being derived ad hoc from the properties of specific tasks or classes of tasks. The distinctions Piaget has drawn between various types of correspondence-transformation relations, and future refinements on these, may turn out to be a useful basis for criteria that generalize across task domains. Thus, for instance, a low response level might be defined as solving a problem by relying on correspondences without cognizing the transformations that are entailed. An intermediate level might involve as solving the

operational aspects of a problem by "bootstrapping" up from correspondences; and a higher level might involve using the relevant operations to generate new correspondences.

Structural theory. Piaget's (e.g., 1970) relational version of structuralism involves the idea of a mutual regulation between parts and wholes at all levels of the cognitive system. Prior to introducing the category-theoretic model, this metatheoretical principle does not appear to have been worked out very explicitly at the level of theory. For instance, the INRC group constitutes the equilibration of the separate concrete operational groupings under a single whole; yet the earlier theory of concrete operations has little if anything to say about processes involved in the mutual assimilation of concrete structures according to laws of this emergent whole. What the theory lacks, at least explicitly, is what a categorical theory of concrete operations would likely have in abundance: means for describing structural correspondences among separate substructures. The theory, in other words, potentially provides a model for the mutual assimilation of substructures, and thereby a model for the generation of emergent laws of the whole structure, and thereby a model for regulatory feedback from the emergent whole to the parts.

It is possible, then, that the new theory would make fairly strong claims about interconceptual synchronies. However, it would be premature to draw any conclusions about such possible claims,

because (a) the details of the theory will be apparent only after much fine-tuning against empirical evidence, and are therefore difficult to predict; and (b) any claims about synchronies or sequences would probably make reference to a set of new measures (as noted above); consequently, their relation to present empirical evidence about synchronies or sequences is indeterminate.

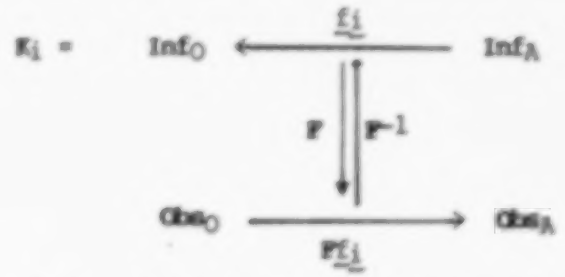
Process theory. One of the most promising aspects of the new formalism is that it may prove adequate for modeling developmental processes as well as cognitive structure. By a developmental process model I mean a theory of how structures generate novelty; an example would be Piaget's (1975/1986) equilibration theory. It is doubtful that any new algebraic formulation that does not provide significant insights into the interaction between structure and process would represent much of an advance over current theories of cognitive development. Of the many issues that must eventually be addressed by a categorical model of developmental process, I will just mention two of the most central: the equilibration of knowledge domains, and the equilibration of more powerful structures. In both cases, describing the generation of novelty involves specifying two types of interaction: the interaction of correspondences with transformations, and the interaction of content knowledge with self-referential knowledge (i.e., knowledge about the activity of the structure itself).

The equilibration of knowledge domains refers to the construction of a stable theory of some content area (*e.g.*, mathematics, morality), including the transformational rules intrinsic to that domain. Intrinsic transformational rules arise in separate domains due to the different possible forms of subject-object interaction (*e.g.*, one does not have the same form of interaction with mathematical objects as one has with moral objects [cf. Turiel & Davidson, 1986]). Similar transformational rules could also arise in separate domains to the extent that similar subject-object interactions are possible. The boundary of a domain is just the range of applicability of its distinct transformational rules.

The equilibration of knowledge structures can be further analyzed as a system of interactions among (a) correspondences between discrete events ("observables") and inferences about their organization, (b) correspondences between discrete actions (upon observables) and inferences about their organization, and (c) correspondences between inferences about the organization of observables and inferences about the organization of actions.

A category-theoretic interpretation of knowledge structures can be approached through the notions of functor and concrete category. A functor is a morphism of categories: given categories **A** and **B**, a functor $F:\mathbf{A}\longrightarrow\mathbf{B}$ assigns to each object a in **A** an object Fa in **B** and to each arrow f in **A** an arrow Ff in **B** (MacLane, 1971, p. 13). A

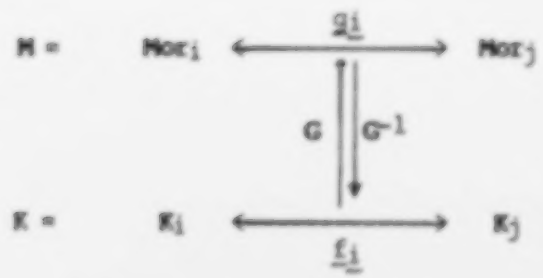
concrete category is a category C equipped with a functor that maps each object of C to an "underlying" set and each arrow of C to a corresponding function defined on those sets (MacLane, 1971, p. 26). A knowledge domain can now be defined as a concrete category K_i:



where Obs_O and Obs_A are sets of observations about events and the subject's actions, respectively; K_i includes inferential structures Inf_O and Inf_A, or theories about these events and actions (i.e., representations of the transformational rules of these events and actions); $\underline{f_i}$ is a morphism preserving these transformational rules; $F\underline{f_i}$ is its image at the level of observables. The diagram suggests an oscillation between the equilibration of correspondences and the equilibration of transformations. The former is an assimilatory process (F) of maximizing similarities between observables and existing theory. The latter is a deductive process ($\underline{f_i}$) of proving that inferences about actions performed with observables are consistent with inferences about how observables are organized. Novelty arises through the introduction of negations in constructing this proof.

As an example, consider a subject given the problem of whether area is conserved inside a closed figure of constant perimeter under transformations of the figure's shape. The problem (Obs_O) is assimilated to an existing commutability schema (Inf_O) with implicit transformational rules by which units of area displaced from one part of the figure are compensated by those added to another part. The subject then attempts a proof by manipulating the figure (Obs_A), leading to the inference (Inf_A) that the same action both maximizes the figure's length in one dimension and minimizes it in the other dimension. This modifies the previous commutability morphism (Inf_O) by introducing a novel parameter, length of side, according to which area goes to zero as the length of any side of the figure is maximized; otherwise commutability applies as a special case.

The central issue in the equilibration of more powerful structures is the transformation of form into content. This can be represented in a category-theoretic approach as the problem of constructing a functor $G: K \longrightarrow M$ where K is a category of distinct knowledge structures (i.e., K includes categories such as K_i in the previous diagram as proper subcategories), and M is a category of morphisms:



In this approach, the translation of form into content is modeled by having objects (Mor_i, Mor_j) at the more abstract level M represent transformations at the less abstract level K. The functor G is regarded as assigning to each morphism of K an object of M. Thus the objects of M are morphisms with their implicit transformational rules. The morphisms of M (*e.g.*, g_i)—which constitute the novel component of the system—are correspondences between these transformational rules. That is, the morphisms g_i are abstractions of the general form of transformations (*e.g.*, joining/separating, adding/subtracting, multiplying/dividing, negating, inverting, etc.) that originated (in K) through interactions with specific concrete events. This is another way of saying that the morphisms of M are the system's reflections on its own activity, or that the novel aspect in a more powerful structure is reflection on the existing structure.

An example of moving from level K to level M would be acquiring a concept of logical validity; that is, being able to determine the validity of an argument with reference to inference rules alone,

rather than with reference to the factual status of premises. In the present model this would be interpreted as follows: imagine a subject who originally has a variety of procedures for evaluating arguments, each specific to a discrete knowledge domain (e.g., the example given above of evaluating an argument about area). Each such procedure is a morphism $\underline{f}_i$ in **K**, and thus potentially an object $\mathbf{Mor}_i$ in **M**. In time the subject reflects on a similarity between two such procedures $\underline{f}_a$, $\underline{f}_b$. This reflecting is the origin of both the the functor **G**, which assigns $\underline{f}_a$ to $\mathbf{Mor}_a$ and $\underline{f}_b$ to $\mathbf{Mor}_b$, and of the morphism $\underline{g}_v$:$\mathbf{Mor}_a \longrightarrow \mathbf{Mor}_b$. (If there are further reflectings on similarities among $\underline{f}_a, \underline{f}_b, \ldots \underline{f}_n$ resulting in morphisms $\underline{g}_a, \underline{g}_b, \ldots \underline{g}_k$, let $\underline{g}_v = \underline{g}_a\underline{g}_b \ldots \underline{g}_k$ [the composite].) Then, in this model $\underline{g}_v$ represents the concept of logical validity.

Conclusion

The category-theoretic model proposed in outline form by Piaget (1977) has so far received little attention in comparison with his earlier work. The empirical and conceptual analyses presented in this paper indicate that the recent model has significant potential for advancing our understanding of cognitive development. The empirical findings support the model's hypothesis that the transition to operational thought is accompanied or preceded by cognition of morphisms, or the construction of correspondences that preserve transformational rules. The conceptual analyses in the

latter part of the paper explore more general issues, extending beyond the realm of concrete operations, in elaborating Piaget's formulation. It appears that an elaborated version of the model could describe both differences in cognitive structure, and processes of developmental change, from infancy through adulthood. This potential integration of formal structure and process models represents a major advance in constructivist theory.

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Table 1

Age Effects for Morphism Performance and Strategy Use

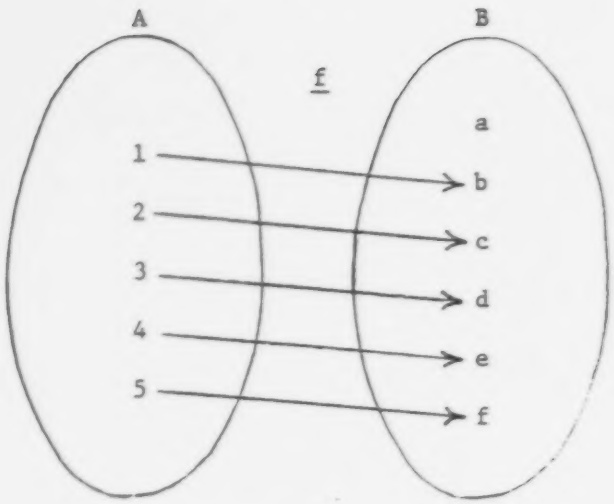
	Age					Age
	5;0-5;5	5;6-5;11	6;0-6;5	6;6-6;11	7;0-7;10	Effect
						$r(70)$
Performance:			Mean score			
Element mapping	14.03	14.04	14.12	14.30	13.86	-.01
Structure mapping	16.03	18.16	18.12	19.57	18.42	.20*
Strategy:			Mean frequency of use			
Trial & error	1.10	1.07	1.12	1.00	1.17	.04
Element oriented	.70	.47	.53	.39	.25	-.18
Structure oriented	.70	.87	.65	.78	.42	-.16
Anticipatory	.50	.60	.71	.83	1.17	.21*

Note. Maximum score for element mapping is 14.67, and for structure mapping, 23.67. (Averaging across three tasks results in the non-integer maxima.)

*$p < .05$.

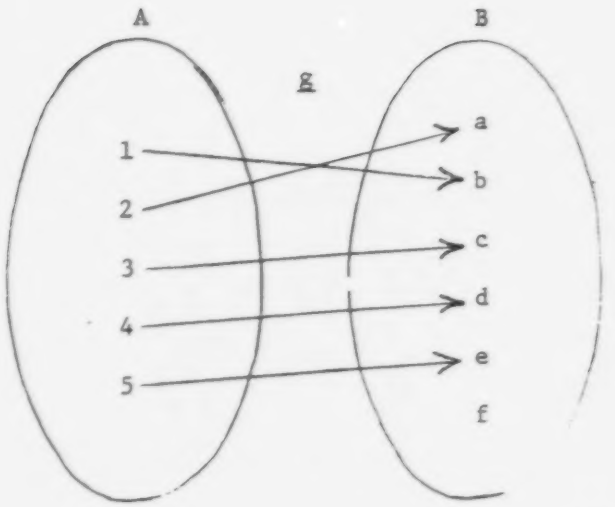
Figure 1. Illustrations of Morphism Properties.

(1)



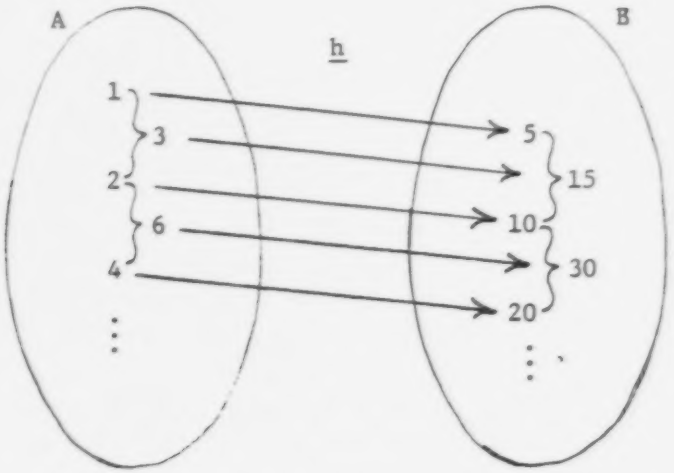
$\langle 1,2 \rangle \in L_1$ $\langle b,c \rangle \in L_2$

(2)



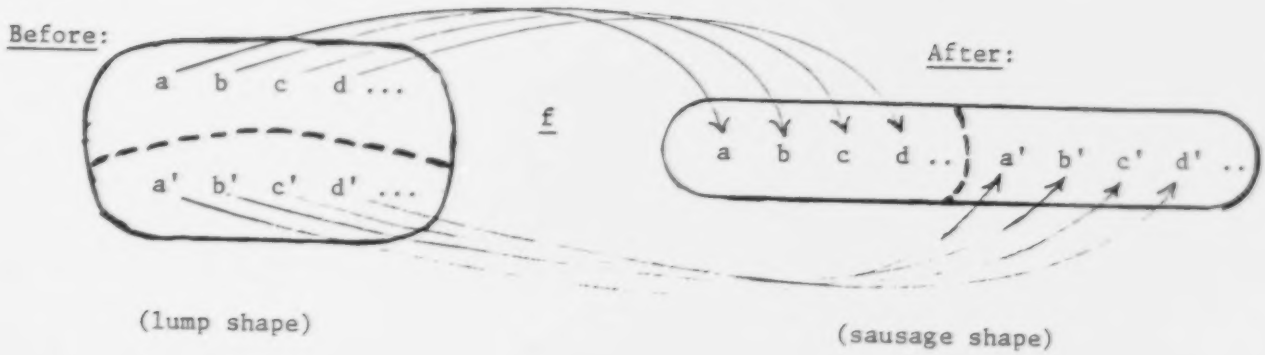
$\langle 1,2 \rangle \in L_1$ $\langle b,a \rangle \notin L_2$

(3)



E.g., $\underline{h}\langle 1 + 2 \rangle = \underline{h}\langle 1 \rangle + \underline{h}\langle 2 \rangle = \underline{h}\langle 3 \rangle = 15$

Figure 2. Conservation of Substance interpreted as a vicariant morphism.

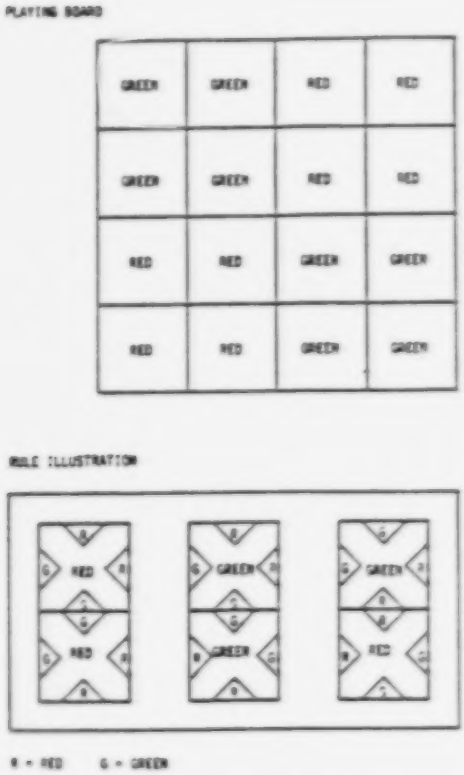


Here, f is a morphism that preserves the quantity of (imagined) subdivisions of the object: the part(s) added to one dimension correspond to the part(s) removed from the reciprocal dimension (cf. commutability).

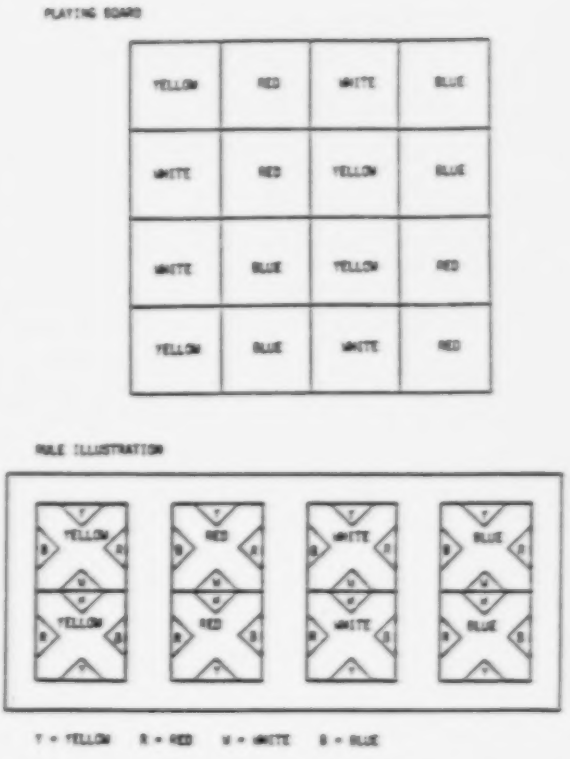
Figure 3.

Materials Used in Morphism Tasks (Rule illustrations also illustrate puzzle pieces.)

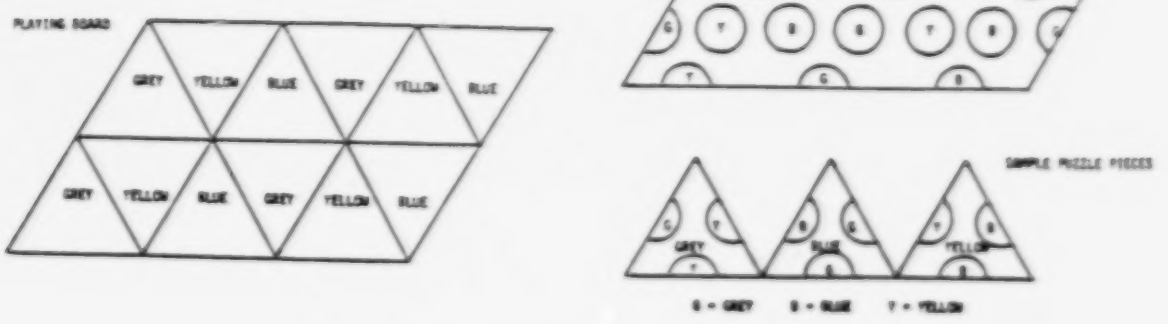
(1)



(2)


PLAYING BOARD
YELLOW RED WHITE BLUE
WHITE RED YELLOW BLUE
WHITE BLUE YELLOW RED
YELLOW BLUE WHITE RED
RULE ILLUSTRATION
Y = YELLOW R = RED W = WHITE B = BLUE

(3)


PLAYING BOARD
GREY YELLOW BLUE GREY YELLOW BLUE
GREY YELLOW BLUE GREY YELLOW BLUE
RULE ILLUSTRATION
SAMPLE PUZZLE PIECES
GREY BLUE YELLOW
G = GREY B = BLUE Y = YELLOW